# ARCHIMEDES' BALANCE AND BIANCHI'S BÄCKLUND TRANSFORMATION FOR QUADRICS

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Abstract. We establish a link between Archimedes' method of integration for calculating areas, volumes and centers of mass of segments of parabolas and quadrics of revolution by factorization via the moments of a balance and an integration technique for a particular integrable system, namely Bianchi's Bäcklund transformation for quadrics.

## Introduction

This paper is organized by first stating the relevant results of Archimedes and Bianchi as they originally appeared, then having a discussion on the notions of Bianchi's result and explaining them in terms of current definitions, providing short motivation and proof for real ruled quadrics for Bianchi's result and then explaining the link between the results of the two authors.

## 1. Archimedes' and Bianchi's results

In *The Method* (lost for 7 centuries and rediscovered in 1906, the year of Bianchi's discovery, so unknown to Bianchi and Lie) as it appears in [5] Archimedes states:

**Theorem 1.1** (The Method)**.** *'... certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.'*

The main theorem of Bianchi's theory of deformations of surfaces applicable to quadrics (which proves the existence of the Bäcklund transformation, its inversion and of the applicability correspondence provided by the Ivory affinity) roughly states:

**Theorem 1.2** (Theorem I)**.** *Every surface $x^0 \subset \mathbb{C}^3$ applicable to a surface $x^0_0 \subseteq x_0$ ($x_0$ being a quadric) appears as a focal surface of a 2-dimensional family of Weingarten congruences, whose other focal surfaces $x^1 = B_z(x^0)$ (called Bäcklund transforms of $x^0$) are applicable, via the Ivory affinity, to surfaces $x^1_0$ in the same quadric $x_0$. The determination of these surfaces requires the integration of a family of Riccati equations depending on the parameter $z$ (ignore for simplicity the dependence on the initial value of the Ricatti equation in the notation $B_z$). Moreover, if we*

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*compose the inverse of the rigid motion provided by the Ivory affinity with the rolling of* $x_0^0$ *on* $x^0$*, then we obtain the rolling of* $x_0^1$ *on* $x^1$ *and* $x^0$ *reveals itself as a* $B_z$ *transform of* $x^1$*.*

## 2. Discussion on the notions appearing in Bianchi's result

*Remark* 2.1. The use of imaginaries (when one complexifies both the real analytic part of the surface (real analytic curves if there is such a family of coordinate curves on the surface or the whole surface if it is real analytic) and the surrounding Euclidean space) is important because it is Lie's interpretation of the Bäcklund transformation for constant Gauß curvature $-1$ surfaces on (imaginary) confocal pseudo-spheres and via integrability of 3-dimensional distributions of *facets* (pairs of point and planes passing through those points; a facet is the infinitesimal version of a surface) that was the tool that allowed Bianchi to prove his result; for this reason we chose to state it in a complex setting.

Except for Lie's influence we shall only work with objects immersed in the Euclidean space

$$(\mathbb{R}^3, < \cdot, \cdot >),\ < x, y >:= x^T y,\ |x|^2 := x^T x \text{ for } x, y \in \mathbb{R}^3.$$

The standard basis $\{e_1, e_2, e_3\}$ satisfies $e_i^T e_j = \delta_{ij}$.

In this setting *'applicable'* surfaces means just 'isometric surfaces' and 'applicability correspondence' means just 'isometric correspondence' ((local) diffeomorphism).

A *'Weingarten congruence'* is a 2-dimensional family of lines on whose two focal surfaces the asymptotic coordinates correspond (equivalently the second fundamental forms of the two focal surfaces are proportional).

*Remark* 2.2. Note that although the correspondence provided by the Weingarten congruence is not the isometric one, a Weingarten congruence is the tool best suited to attack the isometric deformation problem by means of transformation, since it provides correspondence of the characteristics of the isometric deformation problem (according to Darboux these are the asymptotic coordinates) and it is directly linked to the infinitesimal isometric deformation problem (Darboux proved that infinitesimal isometries generate Weingarten congruences and Guichard proved the converse).

Since we want both the *seed* $x^0$ and the *leaves* $x^1 = B_z(x^0)$ to be real surfaces isometric to pieces of real quadrics, the quadric $x_0$ and its *confocal* (with same foci) one $x_z$ must be real doubly ruled, so $x_0, x_z$ are either hyperboloids with one sheet or hyperbolic paraboloids.

The Ivory affinity between confocal quadrics is a natural affine correspondence between confocal quadrics and having good metric properties; thus we have the hyperboloid with one sheet

$$x_z(u,v) := \sqrt{a_1 - z}\frac{1-uv}{u-v}e_1 + \sqrt{z-a_2}\frac{1+uv}{u-v}e_2 + \sqrt{a_3 - z}\frac{u+v}{u-v}e_3,$$
$$a_2 < 0, z < a_1, a_3;\ u, v \in \mathbb{R} \cup \{\infty\},\ u \neq v, \tag{2.1}$$

when the Ivory affinity is given by

$$x_z(u,v) = \sqrt{I_3 - zA}\ x_0(u,v),\ A := \operatorname{diag}[a_1^{-1}\ \ a_2^{-1}\ \ a_3^{-1}]$$

and the hyperbolic paraboloid

$$x_z(u,v) := \sqrt{a_1-z}(u+v)e_1 + \sqrt{z-a_2}(u-v)e_2 + (2uv+\frac{z}{2})e_3,$$
$$a_2<0, z<a_1;\ u,v\in\mathbb{R}, \tag{2.2}$$

when the Ivory affinity is given by

$$x_z(u,v) = \sqrt{I_3 - zA}\ x_0(u,v) + \frac{z}{2}e_3,\ A := \mathrm{diag}[a_1^{-1}\ \ a_2^{-1}\ \ 0].$$

For $D\subseteq\mathbb{R}^2$ domain two isometric surfaces $x_0,\ x: D\to\mathbb{R}^3,\ |dx_0|^2 = |dx|^2$ can be rolled one onto the other: $(x,dx) = (R,t)(x_0,dx_0) := (Rx_0+t, Rdx_0)$,
$(R,t): D\to\mathbf{O}_3(\mathbb{R})\ltimes\mathbb{R}^3$ being a surface in the space of rigid motions (it degenerates to a curve if $x_0,\ x$ are ruled with isometric correspondence of rulings, when the rolling takes place in a 1-dimensional fashion, or to a point if $x_0,\ x$ are rigidly isometric) such that at any instant they meet tangentially and with same differential at the tangency point:

$$dx = Rdx_0. \tag{2.3}$$

Conversely, if $x_0$ can be rolled on $x$ (that is we have (2.3)), then $x_0$ and $x$ are isometric.

For $(u,v)$ parametrization on $D$ and $N_0,\ N$ Gauß maps respectively of $x_0,\ x$ we have $R[x_{0u}\ \ x_{0v}\ \ N_0] = [x_u\ \ x_v\ \ \epsilon N]$, $\epsilon := \pm 1$, so the rotation $R$ of the rolling is uniquely defined (modulo the indeterminacy $\epsilon$) by $R := [x_u\ \ x_v\ \ \epsilon N][x_{0u}\ \ x_{0v}\ \ N_0]^{-1}$; the translation $t$ is then given by $t := x - Rx_0$. The indeterminacy $\epsilon$ decides wether $R$ is special orthogonal or not (we have $N = \frac{x_u\times x_v}{|x_u\times x_v|} = \frac{Rx_{0u}\times Rx_{0v}}{|Rx_{0u}\times Rx_{0v}|} = \det(R)(R^T)^{-1}\frac{x_{0u}\times x_{0v}}{|x_{0u}\times x_{0v}|} = \det(R)RN_0$) and it has a simple geometric explanation: $x_0$ can be rolled on either side of $x$. It is an immediate consequence of (2.3), since (2.3) involves only information about the tangent bundle, so symmetries of the normal bundles (reflections in surfaces) are allowed.

*Remark* 2.3. Although Bianchi was aware of this indeterminacy, its importance seems to have escaped his attention; we shall see later that this indeterminacy provides a simple geometric explanation of an indeterminacy appearing in the rigid motion provided by the Ivory affinity (choice of ruling), which in turn encodes all necessary algebraic information needed to prove Bianchi's result.

*Remark* 2.4. In order to preserve the classical notation (for example $N^Td^2x$ for the second fundamental form of $x$), we shall use the notation $d\wedge$ for exterior (antisymmetric) derivative and $d$ for tensorial (symmetric) derivative; thus $(d\wedge)d = 0$.

For $\omega_1,\ \omega_2$ $\mathbb{R}^3$-valued 1-forms on $D$ and $a,b\in\mathbb{R}^3$, we have $a^T\omega_1\wedge b^T\omega_2 = ((a\times b)\times\omega_1 + b^T\omega_1 a)^T\wedge\omega_2 = (a\times b)^T\omega_1\times\wedge\omega_2 + b^T\omega_1\wedge a^T\omega_2$; in particular

$$a^T\omega\wedge b^T\omega = \frac{1}{2}(a\times b)^T\omega\times\wedge\omega. \tag{2.4}$$

Since both $\times$ and $\wedge$ are skew-symmetric, we have $\omega_1\times\wedge\omega_2 = \omega_1\times\omega_2 + \omega_2\times\omega_1 = \omega_2\times\wedge\omega_1$.

Applying $dx^Td$ to (2.3) we get $dx^Td^2x = dx_0^TR^{-1}dRdx_0 + dx_0^Td^2x_0$; since $dx^Td^2x$ contains only the tangential information of $d^2x$ (namely the Christoffel symbols, or equivalently the Levi-Civita connection), which is the same for both $x_0,\ x$, we have $dx_0^TR^{-1}dRdx_0 = 0$.

For $a \in \mathbb{R}^3$ we have $R^{-1}dRa = R^{-1}dR(a^\perp + a^\top) = a^T N_0 R^{-1}dRN_0 - a^T R^{-1}dRN_0N_0 = \omega_0 \times a,\ \omega_0 := N_0 \times R^{-1}dRN_0$, so under the identification $(\mathbf{o}_3(\mathbb{R}), [,]) \simeq (\mathbb{R}^3, \times)$ we have $R^{-1}dR \simeq \omega_0$. Imposing the compatibility condition $d\wedge$ to $R^{-1}dR$ and to (2.3) we get

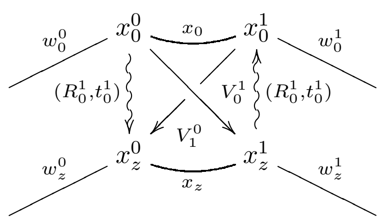
$$d \wedge \omega_0 + \frac{1}{2}\omega_0 \times \wedge\omega_0 = 0,\ \omega_0 \times \wedge dx_0 = 0 \tag{2.5}$$

and thus $\omega_0$ is a flat connection form in $Tx_0$ (it encodes the difference between the Gauß -Codazzi-Mainardi equations for $x, x_0$).

Finally the rigid motion provided by the Ivory affinity exists due to two results of Ivory's and Bianchi's on confocal quadrics (some of the properties used by Bianchi may have already been folklore by that time; for example the change in angles between rulings on confocal hyperbolic paraboloids while preserving their lengths was known to Henrici when he constructed the articulated hyperbolic paraboloid).

**Theorem 2.5** (Ivory)**.** *The orthogonal trajectory of a point on $x_z$ (as $z$ varies) is a conic and the correspondence $x_0 \to x_z$ thus established is affine.*

*This affine transformation (henceforth called the Ivory affinity) preserves the lengths of segments between confocal quadrics: with $V_0^1 := x_z^1 - x_0^0,\ V_1^0 := x_z^0 - x_0^1$ we have $|V_0^1|^2 = |V_1^0|^2$ for pairs of points $(x_0^0, x_z^0),\ (x_0^1, x_z^1)$ corresponding on $(x_0, x_z)$ under the Ivory affinity.*




**Theorem 2.6** (Bianchi I)**.** *If we have the rulings $w_0^0,\ w_0^1$ at the points $x_0^0,\ x_0^1 \in x_0$ and by use of the Ivory affinity we get the rulings $w_z^0,\ w_z^1$ at the points $x_z^0,\ x_z^1 \in x_z$, then $[V_0^1\ \ w_0^0\ \ w_z^1]^T[V_0^1\ \ w_0^0\ \ w_z^1] = [-V_1^0\ \ w_z^0\ \ w_0^1]^T[-V_1^0\ \ w_z^0\ \ w_0^1]$, so there exists a rigid motion $(R_0^1, t_0^1) \in \mathbf{O}_3(\mathbb{R}) \ltimes \mathbb{R}^3$ with*

$$(R_0^1, t_0^1)(x_0^0, x_z^1, w_0^0, w_z^1) = (x_z^0, x_0^1, w_z^0, w_0^1). \tag{2.6}$$

*Moreover $(V_0^1)^T \partial_z|_{z=0} x_z^0 = (V_1^0)^T \partial_z|_{z=0} x_z^1$, so the Ivory affinity has a nice projective property: the symmetry of the tangency configuration*

$$x_z^1 \in T_{x_0^0} x_0 \Leftrightarrow x_z^0 \in T_{x_0^1} x_0. \tag{2.7}$$

*Remark* 2.7. The action of the rigid motion provided by the Ivory affinity resembles a balance; in fact Bianchi uses moments and angles of pairs of lines to explain it.

## 3. Motivation and proof of Bianchi's main theorem

3.1. **The Bianchi-Lie ansatz.** The transformation originally constructed by

Bäcklund in 1883 states that if a constant Gauß curvature $-1$ seed $x$ in general position and an angle $0 < \theta < \frac{\pi}{2}$ are given, then the 3-dimensional distribution formed by facets with centers on circles of radius $\sin\theta$ in tangent planes of $x$ (the circles are themselves centered at the origins of tangent planes), of inclination $\theta$ to these and passing through the origin of these is integrable; moreover the leaves are

constant Gauß curvature $-1$ surfaces and their determination requires the integration of a Ricatti equation.

The Bäcklund transformation when $\theta = \frac{\pi}{2}$ was constructed even earlier (1879, upon some results of Ribaucour from 1870) by Bianchi in his PhD thesis and named the complementary transformation since the seed and the leaf are focal surfaces of a normal Weingarten congruence (and for this type of surfaces the *'complementary'* denomination had already been coined). Thus Bianchi is credited with the idea of using transformations in the study of (isometric deformations of) surfaces, but Bäcklund's merit is the introduction of the spectral parameter.

Lie considered the natural question: because the Bäcklund transformation is of a general nature (independent of the shape of the seed $x$), it must exist (at least as a limiting case) *not* only when $x$ is in general position, *but* also when it coincides with the pseudo-sphere. In this case the 1-dimensional family of non-degenerated leaves (surfaces) degenerates to a 1-dimensional family of degenerated leaves (isotropic rulings on confocal pseudo-sphere) and thus the true nature of the Bäcklund transformation is revealed at the static level of confocal pseudo-spheres. Note that there is no formulation of the Lie ansatz at the level of the sine-Gordon equation, since it deals with imaginary seed and note also that although the Lorentz space of signature $(2, 1)$ is generous enough to contain the pseudo-sphere together with its bundle of tangent planes with positive induced metric, it does not contain the imaginary rulings of the pseudo-sphere.

Thus we consider together with Bianchi ([4],§ 374)

**Theorem 3.1** (Lie's inverting point of view)**.** *The tangent planes to the unit pseudo-sphere $x_0$ cut a confocal pseudo-sphere $x_z$ along circles, thus highlighting a circle in each tangent plane of $x_0$. Each point of the circle, the segment joining it with the origin of the tangent plane and one of the (imaginary) rulings on $x_z$ passing through that point determine a facet. We have thus highlighted a 3-dimensional integrable distribution of facets: its leaves are the ruling families on $x_z$. If we roll the distribution while rolling $x_0$ on an isometric surface $x$ (called seed), it turns out that the integrability condition of the rolled distribution is always satisfied (we have complete integrability), so the integrability of the rolled distribution does not depend on the shape of the seed. The rolled distribution is obtained as follows: each facet of the original distribution corresponds to a point on $x_0$; we act on that facet with the rigid motion of the rolling corresponding to the highlighted point of $x_0$ in order to obtain the corresponding facet of the rolled distribution. The leaves of the rolled distribution (called the Bäcklund transforms of $x$, denoted $B_z(x)$ and whose determination requires the integration of a Ricatti equation) are isometric to the pseudo-sphere. Moreover the seed and any leaf are the focal surfaces of a Weingarten congruence, so the inversion of the Bäcklund transformation has a simple geometric explanation (the seed and leaf exchange places).*

Lie's inverting point of view allows us to call the Bäcklund transformation of constant Gauß curvature $-1$ surfaces *Bäcklund transformation of the pseudo-sphere.*

*Remark* 3.2. Note that the Bäcklund transformation of the pseudo-sphere comes in two flavors, as the facets may reflect in the tangent plane upon which their centers lie, but the complementary transformation comes only in one flavor; this corresponds to a non-degenerate confocal pseudo-sphere having two distinct families

of imaginary rulings and respectively to the two families of rulings degenerating to a single one on the light cone, a singular confocal pseudo-sphere.

*Remark* 3.3. Lie's ansatz is the one susceptible for generalization, since it provides a geometric explanation of the dependence of the Bäcklund transformation on the spectral parameter $z$. Thus if in Lie's interpretation one replaces *'pseudo-sphere'* with *'quadric'* and *'circle'* with *'conic'*, then one gets Bianchi's result except for the Ivory affinity influence.

While looking for the isometric correspondence Bianchi rolled back the seed on the original quadric, in which case the facets of the rolled distribution return to their original location on the confocal quadric $x_z$. Thus if one assumes the isometric correspondence to be valid and of a general nature, it must be independent of the shape of the seed and the answer should be found on the confocal family: the Ivory affinity provides a natural correspondence between confocal quadrics and proving that it provides the isometric correspondence remained a matter of computations.

We are actually able at this point to prove the isometric correspondence provided by the Ivory affinity and the inversion of the Bäcklund transformation for quadrics, assuming **Theorem 2.6** and that the complete integrability of the rolled distribution is checked (we need the leaf $x^1$ to exist): if we roll the seed $x^0$ on $x^0_0$, then the tangent plane of the leaf $x^1$ corresponding to the point of tangency of the rolled $x^0_0$ and the seed $x^0$ will be applied to the facet centered at $x^1_z$ and spanned by $V^1_0,\ x^1_{zu_1}$; further applying the rigid motion $(R^1_0, t^1_0)$ provided by the Ivory affinity it will be applied to $T_{x^1_0}x_0$. In this process $x^1_{u_1}$ is taken to $x^1_{zu_1}$ and further to $x^1_{0u_1}$, so actually $(x^1, dx^1)$ is taken to $(x^1_0, dx^1_0)$; moreover because of the symmetry of the tangency configuration the seed becomes leaf and the leaf becomes seed.

*Remark* 3.4. This geometric argument was the one used by Bianchi to prove the inversion of the Bäcklund transformation, but it seems that he preferred the security of an analytic confirmation to the power of his geometric arguments for the isometric correspondence provided by the Ivory affinity.

3.2. **Proof of Bianchi I.** (which includes also the Ivory theorem).

Note that with $B := 0,\ C := -1$ for (2.1), $B := -e_3,\ C := 0$ for (2.2) and $R_z := I_3 - zA$ both confocal families (2.1) and (2.2) (in fact all confocal families of quadrics) can be implicitly defined by

$$\begin{bmatrix} x_z \\ 1 \end{bmatrix}^T \left( \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} - z \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} x_z \\ 1 \end{bmatrix} = 0,$$

equivalently

$$\begin{bmatrix} x_z \\ 1 \end{bmatrix}^T \begin{bmatrix} AR_z^{-1} & R_z^{-1}B \\ B^T R_z^{-1} & C + zB^T R_z^{-1} B \end{bmatrix} \begin{bmatrix} x_z \\ 1 \end{bmatrix} = 0.$$

From the first definition one can see the metric-projective definition of the family of confocal quadrics: a pencil behavior and Cayley's absolute $\begin{bmatrix} x \\ 0 \end{bmatrix}^T \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = 0,\ x \neq 0$ in the plane at $\infty$ (which encodes the Euclidean structure on $\mathbb{R}^3$).

With $C(z) := (-\frac{1}{2}\int_0^z (\sqrt{R_w})^{-1} dw)B (= 0$ for (2.1) and $= \frac{z}{2}e_3$ for (2.2)) we also have an unifying formula for the Ivory affinity, namely $x_z = \sqrt{R_z}x_0 + C(z)$ and finally it is convenient to work with the normal field $\hat{N}_z := -2\partial_z x_z$ instead of with

the unit normal $N_z$ (note $AC(z)+(I_3-\sqrt{R_z})B=0=(I_3+\sqrt{R_z})C(z)+zB$, since both are 0 for $z=0$ and do not depend on $z$).

Ivory becomes:

- $|V_0^1|^2=|x_0^0+x_0^1-C(z)|^2-2(x_0^0)^T(I_n+\sqrt{R_z})x_0^1+zC=|V_1^0|^2$;

Bianchi I becomes: if $w_0^TAw_0=w_0^T\hat{N}_0=0,\ w_z=\sqrt{R_z}w_0$, etc, then:

- for lengths of rulings: $w_z^Tw_z=|w_0|^2-zw_0^TAw_0=|w_0|^2$;
- for angles between segments and rulings: $(V_0^1)^Tw_0^0+(V_1^0)^Tw_z^0=-z(\hat{N}_0^0)^Tw_0^0=0$;
- for angles between rulings: $(w_0^0)^Tw_z^1=(w_0^0)^T\sqrt{R_z}w_0^1=(w_z^0)^Tw_0^1$;
- for the symmetry of the tangency configuration: $(V_0^1)^T\hat{N}_0^0=(x_0^0)^TA\sqrt{R_z}x_0^1-B^T(x_z^0+x_z^1-C(z))+C=(V_1^0)^T\hat{N}_0^1$.

3.3. **Two algebraic consequences of the tangency configuration.** Note that $x_z(u,v)$ for (2.1) is an affine image of the equilateral hyperboloid with one sheet $H(u,v):=\frac{(1-v^2)e_1+(1+v^2)e_2+2ve_3}{u-v}+\frac{((1-v^2)e_1+(1+v^2)e_2+2ve_3)_v}{2}=\frac{(1-v^2)e_1+(1+v^2)e_2+2ve_3}{u-v}+H(\infty,v)=-H(v,u)$.

Let $\mathcal{B}:=(u-v)^2$ for (2.1) and $\mathcal{B}:=1$ for (2.2), $x_0^0:=x_0(u_0,v_0)$, $x_0^1:=x_0(u_1,v_1)$, $m_0^1:=\mathcal{B}_1x_{zu_1}^1\times V_0^1$ a normal field of the distribution $\mathcal{D}^1$ of facets $\mathcal{F}^1$ passing through $x_z^1$ and spanned by $V_0^1,\ x_{zu_1}^1$ (and similarly $m_0'^1:=\mathcal{B}_1x_{zv_1}^1\times V_0^1$ by considering the other ruling family on $x_z^1$).

For $u_0,v_0,u_1,v_1$ independent variables $\mathcal{B}_1x_{zu_1}^1$ depends only on $v_1$ (quadratically for (2.1) and linearly for (2.2)), so $m_{0u_1}^1=\mathcal{B}_1x_{zu_1}^1\times V_{0u_1}^1=0$ and $m_0^1$ does not depend on $u_1$.

For (2.1) $(\mathcal{B}_1x_{zu_1}^1)_{v_1}\times x_z^1(\infty,v_1)=0$, so $m_0^1=(\mathcal{B}_1x_{zu_1}^1)\times(x_z^1(\infty,v_1)-x_0^0)$ depends quadratically on $v_1$ (the coefficient of the highest order term $v_1^3$ is 0 and that of $v_1^2$ contains $-\frac{1}{2}(\mathcal{B}_1x_{zu_1}^1)_{v_1v_1}\times x_0^0$).

For (2.2) $m_0^1=(\mathcal{B}_1x_{zu_1}^1)\times(x_z^1(0,v_1)-x_0^0)$; since $(\mathcal{B}_1x_{zu_1}^1)_{v_1}\times(x_z^1(0,v_1))_{v_1}\neq 0$, we conclude that $m_0^1$ depends quadratically on $v_1$.

Thus we conclude that in both cases $m_0^1$ depends only on $u_0,v_0,v_1$ and quadratically in $v_1$; this will make the integrability condition (the differential equation subjacent to the Bäcklund transformation) a Ricatti equation in $v_1$.

Henceforth consider only the tangency configuration $(V_0^1)^TN_0^0=0$ (from the proof of Bianchi I this will impose a functional relationship among $u_0,v_0,u_1,v_1$ separately linear in each variable, so a homography is established between them).

If we choose the rulings $w_0^0:=x_{0u_0}^0,\ w_0^1:=x_{0u_1}^1$ respectively at $x_0^0,\ x_0^1$, then we get a rigid motion $(R_0^1,t_0^1)$ provided by the Ivory affinity. If we change the ruling family on $x_0^1$, then the action of the new rigid motion on the facet $T_{x_0^0}x_0$ does not change, so its new rotation must be the old rotation composed with a reflection in $T_{x_0^0}x_0$, because of which the facets $\mathcal{F}^1,\ \mathcal{F}'^1$ reflect in $T_{x_0^0}x_0$ (thus the distributions $\mathcal{D}^1,\ \mathcal{D}'^1$ reflect in $Tx_0$):

$$(x_{zv_1}^1)^T(I_3-2N_0^0(N_0^0)^T)m_0^1=0 \tag{3.1}$$

and $x_{0v_1}^1=R_0^1(I_3-2N_0^0(N_0^0)^T)x_{zv_1}^1$; multiplying this on the left by $(x_{0u_1}^1)^T$ and using the preservation of lengths of rulings under the Ivory affinity we get

$$4(x_{zu_1}^1)^TN_0^0(N_0^0)^Tx_{zv_1}^1du_1dv_1=|dx_z^1|^2-|dx_0^1|^2=-\frac{4z}{\mathcal{B}_1}du_1dv_1. \tag{3.2}$$

Thus we have the next result, essentially due to Bianchi (he uses equivalent computations; in fact most relevant consequences of the tangency configuration are either equivalent to it or to it composed with simple symmetries):

**Lemma 3.5.** *If $x^1_z \in T_{x^0_0}x_0$, then:*

*I (Factorization) The change in the linear element from $x^1_z$ to $x^1_0$ is four times the product of the orthogonal projections of the differentials of the rulings of $x^1_z$ on the normal of $x_0$ at $x^0_0$.*

*II (Reflection) The facets at $x^1_z$ spanned by $V^1_0$ and one of the rulings of $x^1_z$ reflect in $T_{x^0_0}x_0$; therefore the distributions $\mathcal{D}^1$, $\mathcal{D}'^1$ reflect in $Tx^0_0$.*

*Remark* 3.6. Although the equal inclination of facets to tangent planes of seeds from **Remark 3.2** is not preserved when one considers general quadrics instead of pseudo-spheres, the reflection property of facets in tangent planes of seeds remains valid and it is explained by the existence of the rigid motion provided by the Ivory affinity regardless of the choice of rulings. Note however that although this explanation is good enough from an analytic point of view, the choice of ruling still lacks geometric motivation and **Remark 2.3** provides it.

The algebraic relation

$$(N^0_0)^T(2zm^1_0 + m^1_0 \times m^1_{0v_1}) = 0 \tag{3.3}$$

will appear as the total integrability condition. Using (3.1), (3.2) this becomes: $0 = \frac{z(m^1_0)^T x^1_{zv_1}}{(N^0_0)^T x^1_{zv_1}} - \mathcal{B}_1(x^1_{zu_1})^T N^0_0 (V^1_0)^T m^1_{0v_1} = \frac{z(V^1_0)^T(\mathcal{B}_1 x^1_{zv_1} \times x^1_{zu_1} + (\mathcal{B}_1 x^1_{zu_1} \times V^1_0)_{v_1})}{(N^0_0)^T x^1_{zv_1}}$, which is straightforward. Replacing $(m^1_0, v_1)$ with $(m'^1_0, u_1)$ we get a similar relation.

3.4. **Rolling quadrics and distributions.** Let $(R_0, t_0)(x^0_0, dx^0_0) = (x^0, dx^0)$ be the rolling of the piece of quadric $x^0_0 = x_0(u_0, v_0)$ on the isometric surface (seed) $x^0 \subset \mathbb{R}^3$. The facets of the rolled distribution $(R_0, t_0)\mathcal{D}^1$ will become tangent planes to leaves $x^1 := (R_0, t_0)x^1_z = (R_0, x^0)V^1_0$ iff the integrability condition $0 = (R_0 m^1_0)^T dx^1$ holds. We have $R^{-1}_0 dx_1 = d(x^0_0 + V^1_0) + R^{-1}_0 dR_0 V^1_0 = dx^1_z + \omega_0 \times V^1_0$. But $(\omega_0)^\perp = 0$ and $dx^1_z = x^1_{zv_1}dv_1 + x^1_{zu_1}du_1$, so the integrability condition becomes $-(V^1_0)^T \omega_0 \times N^0_0 (m^1_0)^T N^0_0 + (m^1_0)^T x^1_{zv_1} dv_1 = 0$; using (3.1) this becomes $-(V^1_0)^T\omega_0 \times N^0_0 + 2(N^0_0)^T x^1_{zv_1} dv_1 = 0$; multiplying it by $\mathcal{B}_1(N^0_0)^T x^1_{zu_1}$, using (3.2) and $-\mathcal{B}_1(N^0_0)^T x^1_{zu_1} V^1_0 = \mathcal{B}_1(V^1_0 \times x^1_{zu_1}) \times N^0_0 = -m^1_0 \times N^1_0$ we finally get the Ricatti equation:

$$(m^1_0)^T\omega_0 + 2zdv_1 = 0. \tag{3.4}$$

We have $dm^1_0 = m^1_{0v_1}dv_1 + \mathcal{B}_1 dx^0_0 \times x^1_{zu_1}$, so $(dm^1_0)^T \wedge \omega_0 = dv_1 \wedge (m_{0v_1})^T\omega_0$; imposing the total integrability condition $d\wedge$ on (3.4) and using the equation itself we need $-(m^1_0)^T\omega_0 \wedge (m^1_{0v_1})^T\omega_0 + 2z(m^1_0)^T d \wedge \omega_0 = 0$, or, using (2.4) and (2.5): $(N^0_0)^T(2zm^1_0 + m^1_0 \times m^1_{0v_1})(N^0_0)^T\omega_0 \times \wedge\omega_0 = 0$; thus the total integrability is equivalent to (3.3).

## 4. The link to *The Method* of Archimedes

If we roll $x^0_0$ on different sides of the seed $x^0$, then we get the Bäcklund transformation for the other ruling family, so the rolled distributions reflect in the bundle of tangent planes of the seed $x^0$. Thus (3.1) is obtained if one makes the ansatz $x^0_0 = x^0$; the same ansatz for the isometric correspondence provided by the Ivory affinity and the inversion of the Bäcklund transformation (two focal surfaces of a

line congruence are in a symmetric relationship) implies Bianchi's result about the existence of $(R^1_0, t^1_0)$ and the symmetry of the tangency configuration; now (3.2) is obtained as previously described.

While *'certain things first become clear to me'* in *The Method* clearly can be linked to the Bianchi-Lie ansatz, just by fortuitous chance or by Archimedes' clairvoyance (or a combination thereof) the *'by a mechanical method'* meant by Archimedes for the use of balance and slicing corresponds to rolling with its inherent indeterminacy (clearly a mechanical method; in fact the infinitesimal rolling of the seed $x^0$ on the piece of quadric $x^0_0$ corresponds by discretization to the rigid motion provided by the Ivory affinity) and *'although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof'* used by Archimedes for the double reduction ad absurdum corresponds to the geometric arguments involved in the discussion on the rigid motion provided by the Ivory affinity (which may also correspond to Archimedes' geometric identities at the infinitesimal level and using the moments of the balance). Finally *'But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge'* corresponds to the arguments using the flat connection form and the two algebraic consequences of the tangency configuration.

Note that the method at the level of points of facets was known to Bianchi and Lie; however, they had never used the full method, at the level of the planes of the facets too (for this reason the *'of course easier'* ingredient is missing from Bianchi's proofs). Thus if one assumes Theorem I of Bianchi's theory of deformations of quadrics a-priori to be true and to be the metric-projective generalization of Lie's approach, then one naturally *geometrically* gets the necessary algebraic identities needed to prove Theorem I.

For surfaces this method is just a fancy way of reformulating already known identities and which appear naturally enough at the analytic level. But keep in mind that the tangency configuration, (3.1) and (3.2) are equivalent from an analytic point of view and this is not the case in higher dimensions: thus it is very difficult to find the necessary algebraic identities of the static picture from an analytic point of view. Therefore *The Method* of Archimedes, due to its geometric naturalness and the fact that it contains more information, is useful in the study of higher dimensional problems.

Note that although Archimedes and Bianchi-Lie have dealt with different problems, their approach was the same: $P \Rightarrow Q$ with $P$ being either *'The area of a segment of a parabola is an infinite sum of areas of lines'* or *'A line is a deformation of a quadric'*. Such sentences $P$ were not fully accepted as true according to the standard of proof of the times, but they had valid relevant consequences $Q$ which elegantly solved problems not solvable with other methods of those times.

Note that *The Method* of Archimedes went a little closer to the Bäcklund transformation: in an a-priori intuitive elementary non-rigorous geometric argument he transferred all lines (slices of the segment of the parabola) with their centers at the left end of the balance (thus with $\infty$ multiplicity, similarly to the original position of facets in the Bianchi-Lie ansatz). Thus the quadratic mass of a slice of a parabola segment at the left hand side of the balance factorizes in the product of the linear mass of a slice of a triangle and the linear length of the leg of the right hand side of the balance; by integration the mass of the parabola placed with its center at the

left hand side of the balance remains in equilibrium with the mass of the triangle placed with its center at the right hand side of the balance.

Note that in general the facets of the 3-dimensional rolled distribution are differently re-distributed into 2-dimensional families of facets as tangent planes to leaves when the shape of the seed changes (for Bianchi's complementary transformation of the pseudo-sphere the 2-dimensional families of facets as tangent planes to leaves do not change when the shape of the seed changes), but principles and properties independent of the shape of the seed remain valid for facets even in the singular picture: this is Archimedes' contribution to the Bianchi-Lie ansatz and allows us to call this full Bianchi-Lie ansatz *the Archimedes-Bianchi-Lie method*.

Thus one can conclude that *Archimedes' balance for quadrics* is one and the same with *Bianchi's Bäcklund transformation for quadrics* as *principles of a general nature*: they are valid at the infinitesimal level and survive integration and conversely, being principles of a general nature both induce by differentiation and by particular singular configurations the infinitesimal picture where the simplest explanation of these principles reveals itself.

*Remark* 4.1. We have mostly used 'tangent planes' instead of 'tangent spaces' and 'correspondence' instead of '(local) diffeomorphism' since a point, a line or a surface in $\mathbb{R}^3$ come with a 2-dimensional family of tangent planes and either can appear as leaves of integrable distributions. Most of the general statements remain valid when the dimension of the leaves collapses (some of 'surfaces' may have to be replaced with 'lines').

*Remark* 4.2. Note that in [1] Archimedes quotes an even earlier result of Democritos related to the volume of the cone as an analogy and possibly as an inspiration to his method; thus it may be the case that Archimedes followed the same footsteps: from a finite law obtained by empirical observation one gets an infinitesimal law by applying the same finite law to thinner slices of finite objects and a collapsing end process; finally with all relevant information recorded by the infinitesimal objects and which is easier to prove one rigorously proves the general conjectured finite law. This is an inverting point of view similar to Lie's.

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## References

1. Archimedes *The Method*, http://www.gutenberg.org/etext/7825.
2. L. Bianchi *Sur la dèformation des quadriques,* Comptes rendus de l'Académie, **142**, (1906), 562-564, ftp://ftp.bnf.fr/000/N0003096_PDF_562_564.pdf; and **143**, (1906) 633-635.
3. L. Bianchi *Lezioni Di Geometria Differenziale, Teoria delle Transformazioni delle Superficie applicabili sulle quadriche,* Vol **3**, Enrico Spoerri Libraio-Editore, Pisa (1909), http://gallica.bnf.fr/ark:/12148/bpt6k99687j.capture.
4. L. Bianchi *Lezioni Di Geometria Differenziale,* Vol **1-4**, Nicola Zanichelli Editore, Bologna (1922-27) http://quod.lib.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=ABR1998.

5. The MacTutor History of Mathematics archive *Archimedes of Syracuse,* http://www-history.mcs.st-and.ac.uk/Biographies/Archimedes.html.

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